Name: Solutions

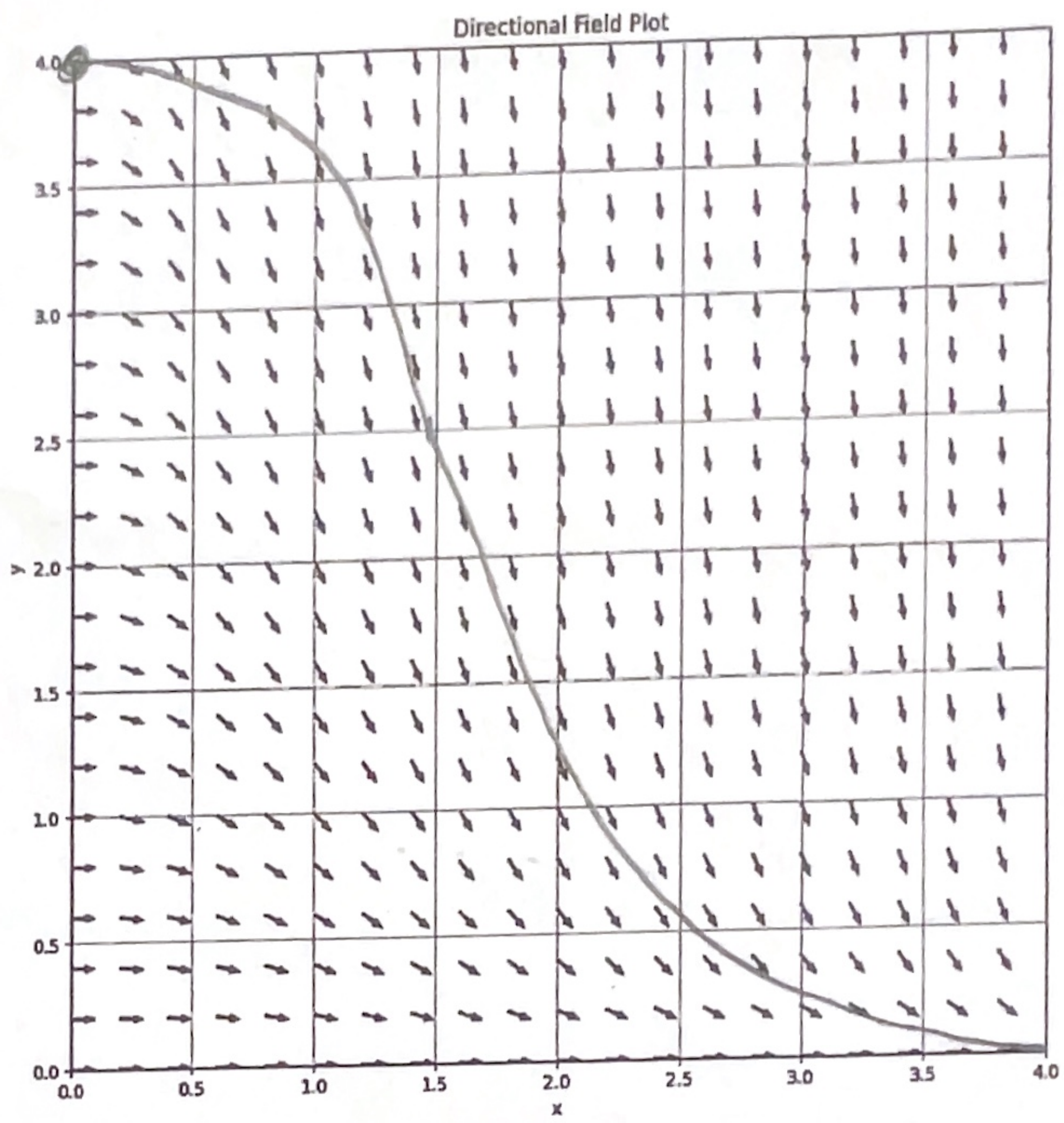
/44

65 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work and use proper notation for full credit. Answers should be in reasonably-simplified form.

1. Consider the DE $y' = -2xy$.

a) *(4 pts)* Its direction field is given. Plot a particular solution satisfying $y(0) = 4$.

b) *(5 pts)* Solve the DE subject to $y(0) = 4$ and find $y(4)$.



$$\frac{dy}{dx} = -2xy \iff \frac{dy}{y} = -2x\,dx$$

$$\ln|y| = -x^2 + \tilde{C}$$

$$|y| = e^{-x^2} \cdot e^{\tilde{C}}$$

$$y = C e^{-x^2}$$

$$y(0) = 4 \Rightarrow C = 4$$

$$y(4) = 4e^{-16}$$

2. *(5 pts)* A tank containing 200 litres of water in which 40 g of salt is dissolved. Pure water is then pumped into the tank at a rate of 4 litres per minute; the well-mixed solution is pumped out at the same rate. Find the number of grams of salt $A(t)$ in the tank at time t and the number of grams $A(\infty)$ of salt in the tank as time $t \to \infty$.

$$\frac{dA}{dt} = 4 \cdot 0 - 4 \cdot \frac{A(t)}{200} \iff A' = -\frac{A}{\cancel{200}\ 50}, \quad A(0) = 40$$

$$\frac{dA}{A} = -\frac{dt}{\cancel{200}\ 50} \iff \ln|A| = -\frac{t}{\cancel{200}\ 50} + C \iff A = e^{-\frac{t}{\cancel{200}\ 50}} \cdot C$$

$$A(0) = 40 \Rightarrow C = 40$$

$$A(t) = 40\, e^{-\frac{t}{\cancel{200}\ 50}}, \quad A(\infty) = 40 \cdot e^{-\infty} = 0.$$

3. *(5 pts)* Solve the linear equation $-y' - y/x = 2 - x$ subject to $y(3) = 2$.

$$y' + \frac{1}{x}y = x - 2$$

$$\mu y' + \mu \frac{1}{x} y = (\mu y)' = \mu y' + \mu' y$$

$$\frac{\mu}{x} = \mu' \Leftrightarrow \frac{d\mu}{\mu} = \frac{dx}{x} \text{ ; } \mu = x \text{ works}$$

$$xy' + y = x^2 - 2x$$

$$(xy)' = x^2 - 2x$$

$$xy = \frac{x^3}{3} - x^2 + c$$

$$y = \frac{x^2}{3} - x + \frac{c}{x}$$

$$y(3) = 3 - 3 + \frac{c}{3} \Rightarrow c = 6$$

$$y(x) = \frac{x^2}{3} - x + \frac{6}{x}$$

4. *(5 pts)* Solve $y'' + 3y = 6 - 2e^{3x}$.

$$\lambda^2 + 3 = 0$$

$$\lambda = \pm i\sqrt{3}$$

$$y_c = c_1 \sin\sqrt{3}x + c_2 \cos\sqrt{3}x$$

$$y_p = A + Be^{3x}$$

plug:

$$9Be^{3x} + 3A + 3Be^{3x} = 6 - 2e^{3x}$$

$$9Be^{3x} + 3Be^{3x} = -2e^{3x}$$

$$12B = -2$$

$$B = -\frac{1}{6}$$

$$A = 2, B = -\frac{1}{6}$$

$$y(x) = c_1 \sin(\sqrt{3}x) + c_2 \cos(\sqrt{3}x) + 2 - \frac{e^{3x}}{6}$$

5. *(5 pts)* Find a suitable form of a particular solution $y_p(x)$ of $y^{IV} + 4y'' = 2 + xe^{2x} + \sin 2x$ (do not evaluate constants).

$\lambda^4 + 4\lambda^2 = 0$, $\lambda^2(\lambda^2+4) = 0$

$y_c = C_1 + C_2 x + C_3 \sin 2x + C_4 \cos 2x$

$y_p = Ax^2 + Bxe^{2x} + Ce^{2x} + Dx\sin 2x + Fx\cos 2x$

$\lambda^2 + 4 = 0 \Leftrightarrow \lambda^2 = -4 \Leftrightarrow \lambda = \pm i\sqrt{4}$

$= \pm 2i$

6. *(5 pts)* Verify that the DE $(2x + ye^{xy})\,dx + (2y + xe^{xy})\,dy = 0$ is exact and find its general solution.

$$\left.\begin{aligned}(2x+ye^{xy})_y &= e^{xy}+yxe^{xy}\\ (2y+xe^{xy})_x &= e^{xy}+yxe^{xy}\end{aligned}\right\} \Rightarrow \text{exact.}$$

$f_x = 2x + ye^{xy}$

$f = x^2 + e^{xy} + g(y)$

$f_y = xe^{xy} + g'(y) = 2y + xe^{xy} \Rightarrow g' = 2y \Rightarrow g = y^2 + c$

$f = x^2 + y^2 + e^{xy} + \tilde{c}$ $\leftarrow$ if you leave the answer in this form, you loose pts.

DE impl. gen. sol:

$$\boxed{x^2 + y^2 + e^{xy} = c}$$

7. a) *(5 pts)* Find the general solution to $y''' - 4y' = 0$.

$\lambda^3 - 4\lambda = 0 \qquad \lambda(\lambda^2 - 4) = 0$

$y(x) = C_1 + C_2 e^{2x} + C_3 e^{-2x}$

b) *(5 pts)* Show that your general solution in part a) is built from a fundamental set of solutions.

$$W = \begin{vmatrix} 1 & e^{2x} & e^{-2x} \\ 0 & 2e^{2x} & -2e^{-2x} \\ 0 & 4e^{2x} & 4e^{-2x} \end{vmatrix} = 2e^{2x} \cdot 4e^{-2x} - 4e^{2x}(-2e^{-2x})$$

$$= 8 + 8 = 16 \neq 0$$

$\Rightarrow$ lin indep

$\Rightarrow$ fund. set.

Extra credit. *(4 pts)* Draw the phase portrait of the autonomous DE $y' = \cos^2 y + \sin 2y + \sin^2 y$ and classify all critical points.

$y' = \cos^2 y + \sin 2y + \sin^2 y = \cos^2 y + 2\sin y \cos y + \sin^2 y$

$= (\cos y + \sin y)^2 = 2\left(\frac{1}{\sqrt{2}}\cos y + \frac{1}{\sqrt{2}}\sin y\right)^2$

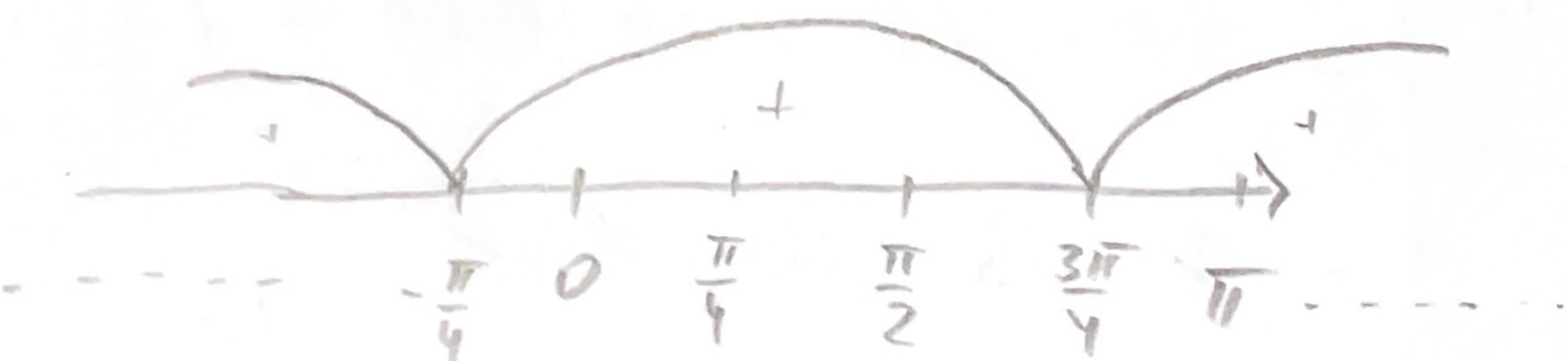
$= 2\sin^2\left(y + \frac{\pi}{4}\right)$

roots: $y + \frac{\pi}{4} = k\pi$, $y = k\pi - \frac{\pi}{4}$, $k \in \mathbb{Z}$

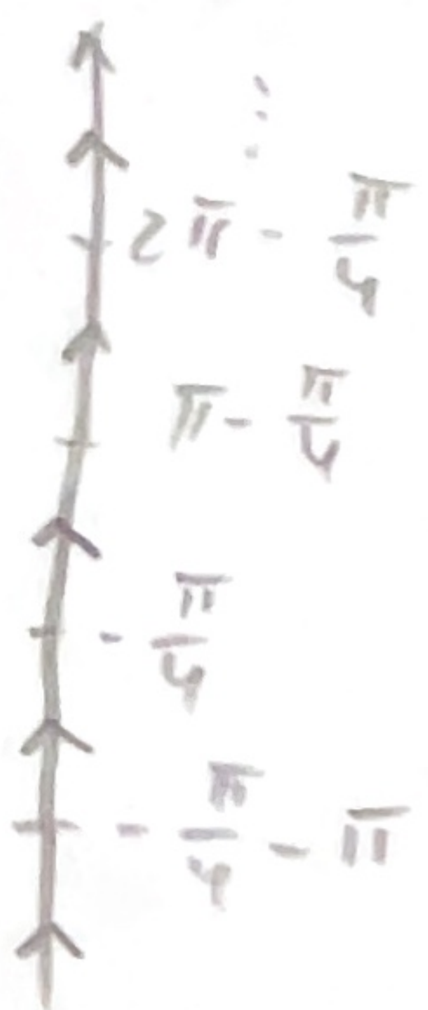
Note: $y' \geq 0$

EXTRA SPACE

approx. graph $y = 2\sin^2\left(x + \frac{\pi}{4}\right)$



So $y = k\pi - \frac{\pi}{4}$ - semistable points ($k \in \mathbb{Z}$)



← phase portrait.